1. (73 I #3)

For any real $2 \times 2$ matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ we define $Tr(A)$ to be the sum of the diagonal elements of $A$.

(a) Show that, for any real $2 \times 2$ matrix $A$, $B$ and any real number $t$,

$Tr(A + B) = Tr(A) + Tr(B)$

$Tr(tA) = tTr(A)$

$Tr(AB) = Tr(BA)$

Hence or otherwise, prove that, for any real $2 \times 2$ matrix $A$ and $B$, $AB - BA \neq I$ where $I$ is the $2 \times 2$ identity matrix.

(b) Omitted

2. (73 I #5)

Let $V$ be the set of all real $3 \times 1$ matrices $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ under matrix addition and scalar multiplication, and let $A$ be any fixed real $3 \times 3$ matrix. A mapping $\phi : V \rightarrow V$ is given by:

$\phi(x) = Ax$ for every $x$ in $V$.

(a) Show that for all $x, y \in V$ and for all real numbers $\lambda$, $\phi(x + y) = \phi(x) + \phi(y)$ and $\phi(\lambda x) = \lambda \phi(x)$

(b) Let $K$ be the set of all those vectors of $V$ which are mapped by $\phi$ to the zero vector $\mathbf{0}$ of $V$. Show that $0 \in K$ and that if $x, y \in K$ and $\lambda$ is any real number, then $x + y \in K$ and $\lambda x \in K$.

(c) If $A = \begin{pmatrix} 1 & 2 & -1 \\ 1 & -1 & 1 \\ 4 & -1 & 2 \end{pmatrix}$, determine the value of the real number $a$ so that the vector

$\begin{pmatrix} 0 \\ a \\ 5a \end{pmatrix} = \phi(x)$ for some $x$ in $V$, and find all vectors $x$ which are mapped to $\begin{pmatrix} 0 \\ a \\ 5a \end{pmatrix}$ when $a$

assumes this value.
3. (76 I #4)

Let \( p \) \((p \neq 0,1)\) be a real number, and \( A = \begin{pmatrix} 2p - p^2 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \). Let \( x_0 = 0 \) and \( x_n = \sum_{k=1}^{n} kp^{k-1} \) for all positive integers \( n \).

(a) Prove that \( A^n = \begin{pmatrix} (n+1)p^n & -np^{n+1} & x_n \\ np^{n-1} & -(n-1)p^n & x_{n-1} \\ 0 & 0 & 1 \end{pmatrix} \) for all positive integers \( n \).

(b) Let \( y_0 = a \), \( y_1 = b \) and \( y_{n+1} = 2py_n - p^2y_{n-1} + 1 \) for all positive integers \( n \). Verify that

\[
\begin{pmatrix} y_{n+1} \\ y_n \\ 1 \end{pmatrix} = A \begin{pmatrix} y_n \\ y_{n-1} \\ 1 \end{pmatrix}.
\]

Hence show that \( y_n = bn^{n-1} - a(n-1)p^n + x_{n-1} \) for all positive integers \( n \), and verify that \( x_{n-1} = \frac{1-np^{n-1}+(n-1)p^n}{(1-p)^2} \).

4. (77 I #4)

Let \( M \) be the set of real, non-singular \( 2 \times 2 \) matrices under the operation of matrix multiplication. A mapping \( d : M \rightarrow \mathbb{R} \setminus \{0\} \) of \( M \) into the set of all non-zero real numbers is defined as follows: for any \( A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \) in \( M \), \( d(A) = a_{11}a_{22} - a_{12}a_{21} \).

Show that

(a) \( d(AB) = d(A)d(B) \) for any \( A, B \) in \( M \);
(b) The mapping \( d \) is surjective but not injective;
(c) Omitted
(d) Omitted

5. (76 I #6)

(a) Let \( x \) be a non-zero real number greater than -1. Prove by induction that for any positive integer \( n \) greater than 1, \( (1 + x)^n > 1 + nx \).

(b) Let \( t \) be any fixed positive number. Consider the sequence \( a_1, a_2, a_3, \cdots \) where \( a_n = \sqrt[n]{t} \), the positive \( n \)-th root of \( t \).

(i) Let \( t > 1 \), show that \( \sqrt[n]{t} > 1 \).
Putting $\sqrt[n]{t} = 1 + x_n$ and using (a), or otherwise, show that $1 < \sqrt[n]{t} < 1 + \frac{t-1}{n}$ for $n \geq 2$.

Hence show that for $t > 1$, $\lim_{n \to \infty} a_n = 1$.

(ii) Show that for all $t > 0$, $\lim_{n \to \infty} a_n = 1$.

6. (79 II #7)

(a) (i) For any $x \geq 0$, show that $(1 + x)^n \geq \frac{n(n-1)}{2} x^2$ for any positive integer $n$. by putting $x = \sqrt[n]{n} - 1$ in the above inequality, or otherwise, show that $\lim_{n \to \infty} \sqrt[n]{n} = 1$.

(ii) Evaluate the expression $\lim_{n \to \infty} \sqrt[n]{n^3 + n + 1} \left/ \left( n^2 + 1 \right) \right.$

(b) Find the absolute maximum of the function $f(x) = x^\frac{1}{3}$ on $[1, \infty)$. Hence, or otherwise, find the greatest value among the sequence $\left\{ \sqrt{n} \right\}_{n=1,2,3,\ldots}$ (It is known that $2 < e < 3$).

7. (80 I #4)

(a) The terms of a sequence $y_1, y_2, y_3, \cdots$
satisfy the relation $y_k = Ay_{k-1} + B \ (k \geq 2)$
where $A, B$ are constants independent of $k$ and $A \neq 1$.
Guess an expression for $y_k \ (k \geq 2)$ in terms of $y_1, A, B$ and $k$ and prove it.

(b) The terms of a sequence $x_0, x_1, x_2, \cdots$
satisfy the relation $x_k = (a+b)x_{k-1} + abx_{k-2} \ (k \geq 2)$
where $a, b$ are non-zero constants independent of $k$ and $a \neq b$.
(i) Express $x_k - ax_{k-1} \ (k \geq 2)$ in terms of $x_1 - ax_0, b$ and $k$.
(ii) Using (a), or otherwise, express $x_k \ (k \geq 2)$ in terms of $x_0, x_1, a, b$ and $k$.

(c) If the terms of the sequence $x_0, x_1, x_2, \cdots$
satisfy the relation $x_k = \frac{1}{3} x_{k-1} + \frac{2}{3} x_{k-2} \ (k \geq 2)$,
express $\lim_{k \to \infty} x_k$ in terms of $x_0$ and $x_1$. 
8. (81 I #2)

Let \( \{a_n\} \) be a sequence of real numbers. It is known that, if

(i) \( \{a_n\} \) satisfies \( a_n \leq a_{n+1} \) for all \( n \), and;

(ii) there exists a real number \( K \) such that \( a_n \leq K \) for all \( n \),

then \( \{a_n\} \) converges.

(a) Show that a sequence \( \{b_n\} \) of real numbers is convergent if

(i) \( b_n \geq b_{n+1} \) for all \( n \), and;

(ii) there exists a real number \( M \) such that \( b_n \geq M \) for all \( n \),

(b) Given two positive real numbers \( a \) and \( b \) such that \( a < b \). Let \( \{x_n\} \) and \( \{y_n\} \) be two sequences satisfying \( x_1 = a, \ y_1 = b, \) and

\[
x_{n+1} = \sqrt{x_n y_n}, \quad y_{n+1} = \frac{x_n + y_n}{2}
\]

for all positive integers \( n \). Show that both \( \{x_n\} \) and \( \{y_n\} \) converge and \( \lim_{n \to \infty} x_n = \lim_{n \to \infty} y_n. \)